

# PERIODIC SOLUTIONS OF PERIODICALLY PERTURBED PLANAR AUTONOMOUS SYSTEMS: A TOPOLOGICAL APPROACH <sup>†</sup>

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## Abstract

Aim of this paper is to investigate the existence of periodic solutions of a nonlinear planar autonomous system having a limit cycle  $x_0$  of least period  $T_0 > 0$  when it is perturbed by a small parameter,  $T_1$ -periodic, perturbation. In the case when  $T_0/T_1$  is a rational number  $l/k$ , with  $l, k$  prime numbers, we provide conditions to guarantee, for the parameter perturbation  $\varepsilon > 0$  sufficiently small, the existence of  $klT_0$ -periodic solutions  $x_\varepsilon$  of the perturbed system which converge to the trajectory  $\tilde{x}_0$  of the limit cycle as  $\varepsilon \rightarrow 0$ . Moreover, we state conditions under which  $T = klT_0$  is the least period of the periodic solutions  $x_\varepsilon$ . We also suggest a simple criterion which ensures that these conditions are verified. Finally, in the case when  $T_0/T_1$  is an irrational number we show the nonexistence, whenever  $T > 0$  and  $\varepsilon > 0$ , of  $T$ -periodic solutions  $x_\varepsilon$  of the perturbed system converging to  $\tilde{x}_0$ . The employed methods are based on the topological degree theory.

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## 1 Introduction

The paper is devoted to the study of the existence and of the behaviour as  $\varepsilon \rightarrow 0$  of periodic solutions of a perturbed autonomous differential system in  $\mathbb{R}^2$  of the form

$$\dot{x} = \psi(x) + \varepsilon\phi(t, x), \quad (1)$$

where  $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a twice continuously differentiable function,  $\phi : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is continuous and  $T_1$ -periodic with respect to the first variable and  $\varepsilon > 0$  is a small parameter.

To be specific, we assume that at  $\varepsilon = 0$  the autonomous system

$$\dot{x} = \psi(x), \quad (2)$$

has a limit cycle  $x_0 = x_0(t)$ ,  $t \geq 0$ , of least period  $T_0$  satisfying the following condition

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(A<sub>0</sub>)- the linear system

$$\dot{y} = \psi'(x_0(t))y \quad (3)$$

does not have  $2T_0$ -periodic solutions linearly independent with  $\dot{x}_0(t)$ .

Furthermore, we assume

(A<sub>1</sub>)- there exist  $l, k \in \mathbb{N}$  such that  $\frac{T_0}{T_1} = \frac{l}{k}$ , with  $l$  and  $k$  prime numbers.

This paper addresses the following problems:

1. *To provide conditions on the function  $\phi$  which guarantee the existence of  $\varepsilon_* > 0$  such that system (1) has a  $T$ -periodic solution  $x_\varepsilon$ , with  $T = T(T_0, T_1)$  and  $\varepsilon \in (0, \varepsilon_*)$  satisfying the property*

$$x_\varepsilon(t) \rightarrow \tilde{x}_0 \text{ as } \varepsilon \rightarrow 0, \quad \text{whenever } t \in [0, T],$$

where  $\tilde{x}_0 = \{x \in \mathbb{R}^2 : x = x_0(t), t \in [0, T_0]\}$  is the trajectory of the limit cycle of (2).

2. *To find an explicit estimation of  $\varepsilon_* > 0$ .*

3. *To investigate the existence of periodic solutions for system (1) in the case when condition (A<sub>1</sub>) is not verified, i.e. when  $\frac{T_0}{T_1}$  is an irrational number.*

Several contributions have been made toward solving the problem of the existence and the nature of periodic solutions of a periodically perturbed autonomous system which are close to the periodic solution of the autonomous system. We refer to the papers [20], [21], [19] and to the memoir [22] for first and second order perturbed autonomous systems respectively. The employed methods are based on perturbation and implicit function techniques which require more regularity on the function  $\phi$  of that required in this paper. In [20], under assumptions similar to (A<sub>0</sub>) and (A<sub>1</sub>) the author provides sufficient conditions for the existence and stability of periodic solutions of a perturbed autonomous system in  $\mathbb{R}^n$ . We would like to point out that in this paper we are interested in periodic solutions of general form, that is without any condition on their rate of convergence to  $\tilde{x}_0$  with respect to  $\varepsilon$ . In such a generality the uniqueness of the periodic solutions is not guaranteed as in [20], where a special class of periodic solutions, converging to  $\tilde{x}_0$  at the same rate of  $\varepsilon$ , is considered. Similar results for the existence of  $D$ -periodic functions, namely for functions whose derivative is periodic, can be found in [9] and [10]. Furthermore, in [3], [4] and [5] analogous existence and stability results are established under different assumptions. Specifically, in [3], under the assumption that 1 is the only characteristic multiplier of the linearized system, which is a  $q$ -th root of unity, for  $q \in \mathbb{N}$ , it is proved that for any  $p \in \mathbb{N}$  there exists a parametrized family of periods and corresponding periodic solutions of the perturbed autonomous system such that at  $\varepsilon = 0$  the period is given by  $qT_0/p$ . In [4] and [5] the behaviour, with respect to the period of the disturbance, of the periodic surface described by a functional equation and associated to the perturbed autonomous system is studied. Finally, in [11] and [12] existence and multiplicity results of periodic solutions of a periodically perturbed autonomous second order equation, defined on a manifold, were proved by means of topological methods. An example of periodically perturbed autonomous systems arising in Mathematical Biology is given in [13].

An other approach to the study of the existence of periodic solutions of perturbed autonomous systems based on topological degree and index theory is presented in [8] and [1] respectively. In these papers

the perturbation is not necessarily small, in the sense that it does not contain small perturbation parameters, the period  $T$  is fixed and the autonomous system may have periodic orbits of period less than  $T$ .

In this paper to solve the proposed problems we use a different approach which was previously introduced and successfully employed by the authors in [14], [15] and [16] to treat the existence problem of periodic solutions of perturbed nonautonomous systems of prescribed period. Specifically, in [14] we consider the nonautonomous system of differential equations described by

$$\dot{x} = \psi(t, x) + \varepsilon \phi(t, x), \quad (NA)$$

where  $\phi, \psi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  are continuously differentiable,  $T$ -periodic with respect to time  $t$ , functions and  $\varepsilon$  is a small positive parameter.

The proposed approach is based on the linearized system associated to (NA)

$$\dot{y} = \frac{\partial \psi}{\partial x}(t, \Omega(t, 0, \xi))y + \phi(t, \Omega(t, 0, \xi)), \quad (LNA)$$

where  $\xi \in \mathbb{R}^n$  and  $\Omega(\cdot, t_0, \xi)$  denotes the solution of (NA) at  $\varepsilon = 0$  satisfying  $x(t_0) = \xi$ . Specifically, consider the change of variable

$$z(t) = \Omega(0, t, x(t)), \quad (CV)$$

and the solution  $\eta(\cdot, s, \xi)$  of (LNA) such that  $y(s) = 0$ . If there exists a bounded open set  $U \subset \mathbb{R}^n$  such that  $\Omega(T, 0, \xi) = \xi$  for any  $\xi \in \partial U$ , and  $\eta(T, s, \xi) - \eta(0, s, \xi) \neq 0$ , for any  $s \in [0, T]$ , and any  $\xi \in \partial U$ . Then (NA) has a  $T$ -periodic solution for  $\varepsilon > 0$  sufficiently small provided that  $\Gamma(\eta(T, 0, \cdot), U) \neq 0$ . Here  $\Gamma(F, U)$  denotes the rotation number of a continuous map  $F : \overline{U} \rightarrow \mathbb{R}^2$ . Observe that  $\Gamma(F, U)$  coincides with  $\deg(F, U, 0)$ , the topological degree of  $F$  at 0 relative to  $U$ . In what follows, we will omit the point 0 in the notation for the topological degree.

The advantage of the proposed approach as compared with the classical averaging method, which is one of the most useful tools for treating the existence problem of periodic solutions of nonautonomous periodic systems, mainly consists in the fact that in order to use this second method for establishing the existence of periodic solutions in perturbed systems of the form (NA) one must assume that the change of variable (CV) is  $T$ -periodic with respect to  $t$  for every  $T$ -periodic function  $x$  such that  $\Omega(0, t, x(t)) \in U$ , for any  $t \in [0, T]$ , instead that only on the boundary of the bounded open set  $U$ .

The same assumption is necessary in vibrational control problems, [2] and [23], to reduce the considered system to the standard form for applying the averaging method. For an extensive list of references on this topic see [7].

Our approach has been also employed in [15] to prove the existence of periodic solutions for a class of first order singularly perturbed differential systems.

Furthermore, in [16] we have considered two nonlinear small periodic perturbations of  $\psi(t, x)$  with multiplicative different powers of  $\varepsilon > 0$  and we have proved the existence of  $T$ -periodic solutions of the resulting system. We have also showed the presence of the so-called frequency pulling phenomenon in the case of a special class of planar autonomous systems when they are perturbed by periodic terms with period close to that of the periodic solution of the autonomous system, whose existence is assumed. The aim of this paper is at solving problems 1-3 by using the approach outlined above, extending in this way its application to the study of the existence of periodic solutions and of their behaviour for general, periodically perturbed, planar autonomous systems having a limit cycle. In particular, we

are interested in investigating the relationship between the period of the limit cycle and that of the nonautonomous perturbations in order to have periodic solutions of (2) with the period expressed in terms of the previous ones.

To the best knowledge of the authors the present paper treats for the first time the problem of evaluating  $\varepsilon_* > 0$ , i.e. problem 2. stated before, for the application of the averaging principle via topological methods.

The paper is organized as follows. In section 2 we show our main result: theorem 1 which provides sufficient conditions to ensure, for  $\varepsilon > 0$  sufficiently small, the existence of  $klT_0$ –periodic solutions  $x_\varepsilon$  to (2) converging to  $\tilde{x}_0$  as  $\varepsilon \rightarrow 0$ . In theorem 2 we show, under a mild extra assumption, that  $klT_0$  is the least period.

In section 3 we present a simple method to ensure that the conditions of theorem 1 are satisfied.

Finally, in Section 4 we establish a nonexistence result: theorem 4, in the case when  $T_0/T_1$  is irrational. Precisely, we show that, for any  $T > 0$  and  $\varepsilon > 0$ , there is not a  $T$ –periodic solution  $x_\varepsilon$  of (2) such that  $x_\varepsilon \rightarrow \tilde{x}_0$  as  $\varepsilon \rightarrow 0$ . An example illustrating the existence result is also provided.

## 2 The main result

Through this section we assume that  $T = klT_0$  and we will denote by  $F'_{(i)}$  the derivative of the function  $F$  with respect to the  $i$ -th argument. Let  $x(t) = \Omega(t, 0, \xi)$  be the solution of system (2) satisfying  $x(0) = \xi$ . Consider the following auxiliary system of linear ordinary differential equations

$$\dot{y} = \psi'(\Omega(t, 0, \xi))y + \phi(t, \Omega(t, 0, \xi)). \quad (4)$$

and let  $y(t) = \eta(t, s, \xi)$  the solution of (4) satisfying  $y(s) = 0$ . Observe that if  $\psi = 0$  then

$$\eta(T, s, \xi) - \eta(0, s, \xi) = \int_0^T \phi(\tau, \xi) d\tau.$$

Therefore, in this case, the function  $\frac{1}{T}(\eta(T, s, \xi) - \eta(0, s, \xi))$  is the average on the interval  $[0, T]$  of the function  $\phi$  with respect to the first variable. This function is the basis of the classical averaging method, one of the most relevant tools to investigate the existence of periodic solution of (NA) when  $\psi = 0$  (see [6]).

We can prove the following preliminary result.

### Lemma 1

$$\eta(t, s, \xi) = \Omega'_{(3)}(t, 0, \xi) \int_s^t \Phi(\tau, \xi) d\tau,$$

where

$$\Phi(t, \xi) = \Omega'_{(3)}(0, t, \Omega(t, 0, \xi))\phi(t, \Omega(t, 0, \xi)). \quad (5)$$

**Proof:** Observe that the matrix  $\Omega'_{(3)}(t, 0, \xi)$  is the fundamental matrix, satisfying  $\Omega'_{(3)}(0, 0, \xi) = I$ , for the linear system

$$\dot{y} = \psi'(\Omega(t, 0, \xi))y.$$

Furthermore,  $\left(\Omega'_{(3)}(t, 0, \xi)\right)^{-1} = \Omega'_{(3)}(0, t, \Omega(t, 0, \xi))$ , in fact, by deriving with respect to  $\xi$  the identity

$$\Omega(0, t, \Omega(t, 0, \xi)) = \xi, \text{ whenever } \xi \in \mathbb{R}^2,$$

we obtain

$$\Omega'_{(3)}(0, t, \Omega(t, 0, \xi))\Omega'_{(3)}(t, 0, \xi) = I, \text{ whenever } \xi \in \mathbb{R}^2. \quad (6)$$

Therefore, by the variation of constants formula for the nonhomogeneous system (4) we obtain

$$\begin{aligned} \eta(t, s, \xi) &= \int_s^t \Omega'_{(3)}(t, 0, \xi) \left(\Omega'_{(3)}(\tau, 0, \xi)\right)^{-1} \phi(\tau, \Omega(\tau, 0, \xi)) d\tau = \\ &= \Omega'_{(3)}(t, 0, \xi) \int_s^t \Phi(\tau, \xi) d\tau. \end{aligned}$$

□

Since the trajectory  $\tilde{x}_0$  is a Jordan curve, its interior  $U$  is a bounded, simply connected, open set of  $\mathbb{R}^2$ . In order to prove, by the proposed approach, the existence of periodic solutions to (1) converging to  $\tilde{x}_0$  as  $\varepsilon \rightarrow 0$ , we introduce a family of open sets  $W_\gamma(U)$  as follows

$$W_\gamma(U) = \begin{cases} U \setminus (\partial U + |\gamma|B) & \text{if } \gamma < 0, \\ U \cup (\partial U + |\gamma|B) & \text{if } \gamma > 0, \end{cases}$$

where  $B \subset \mathbb{R}^2$  is the open unit ball, thus we have

$$B_\gamma(\partial U) := ((W_\gamma(U) \cup U) \setminus W_\gamma(U)) \bigcup ((W_\gamma(U) \cup U) \setminus U) \rightarrow \tilde{x}_0 \quad \text{as } \gamma \rightarrow 0, \quad (7)$$

where  $\partial U = \tilde{x}_0$ .

**Definition 1** Define the following positive constants

$$\begin{aligned} M_\gamma &= \max_{t \in [0, T], \xi \in B_\gamma(\partial U)} \|\Phi(t, \xi)\|, \\ M'_\gamma &= \max_{t \in [0, T], \xi \in B_\gamma(\partial U)} \|\Phi'_{(2)}(t, \xi)\|, \\ L'_\gamma &= \max_{\xi \in B_\gamma(\partial U)} \|\Omega'_{(3)}(T, 0, \xi)\|, \\ L''_\gamma &= \max_{\xi \in B_\gamma(\partial U)} \|\Omega''_{(3)(3)}(T, 0, \xi)\|, \\ K_0 &= \min_{s \in [0, T], \xi \in \partial U} \|\eta(T, s, \xi) - \eta(0, s, \xi)\|, \\ K_\gamma &= \min_{\xi \in \partial W_\gamma(U)} \|\xi - \Omega(T, 0, \xi)\|. \end{aligned}$$

Moreover, let  $\gamma_0 > 0$  such that system (2) does not have  $T_0$ -periodic solutions with initial condition belonging to the open set  $B_{-\gamma_0}(\partial U) \cup B_{\gamma_0}(\partial U)$ .

Observe that condition  $(A_0)$  guarantees the existence of the constant  $\gamma_0$  in the previous definition.

We can now formulate the main result of the paper.

**Theorem 1** *Assume  $(A_0)$ ,  $(A_1)$  and*

- $(A_2)$ -  $\eta(T, s, \xi) - \eta(0, s, \xi) \neq 0$ , for any  $s \in [0, T]$ , and any  $\xi \in \partial U$ ,
- $(A_3)$ -  $\deg(\eta(0, T, \cdot), U) \neq 1$ .

*Then for every  $-\gamma_0 < \gamma < \gamma_0$ ,  $\gamma \neq 0$  and for*

$$0 < \varepsilon < \min \left\{ \frac{K_0}{T^2 M_\gamma (M'_\gamma + \sqrt{2} M_\gamma L''_\gamma + M'_\gamma L'_\gamma)}, \frac{K_\gamma}{T M_\gamma (1 + L'_\gamma)} \right\} := \varepsilon_\gamma \quad (8)$$

*system (1) has a  $T$ -periodic solution  $x_\varepsilon$  belonging to the set*

$$\left\{ x \in \mathbb{R}^2 : x = \Omega(t, 0, \xi), t \in [0, T], \xi \in B_\gamma(\partial U) \right\}.$$

To prove this theorem we need the following lemma.

**Lemma 2** *Assume  $(A_0)$ . Then there exists  $\gamma_1 > 0$  such that for  $-\gamma_1 < \gamma < \gamma_1$  and  $\gamma \neq 0$  we have*

$$\deg(I - \Omega(T, 0, \cdot), W_\gamma(U)) = 1.$$

**Proof:** Condition  $(A_0)$  ensures that the characteristic multiplier of  $\Omega'_{(3)}(T, 0, x(0))$  different from  $+1$  is not equal to  $-1$ . Therefore the limit cycle  $x_0$  is either asymptotically stable or unstable, thus in a sufficiently small neighborhood of  $\tilde{x}_0$  the map  $\xi \rightarrow \Omega(T, 0, \xi)$  does not have fixed points different from those belonging to  $\tilde{x}_0$ . In particular,

$$\xi \neq \Omega(T, 0, \xi) \quad \text{for any } \xi \in \partial W_\gamma(U) \quad (9)$$

and  $|\gamma| \neq 0$  sufficiently small. Hence by Corollary 2 of [8] we have that

$$\deg(I - \Omega(T, 0, \cdot), W_\gamma(U)) = \deg(\psi, W_\gamma(U)),$$

On the other hand from (9) we have that

$$\deg(\psi, W_\gamma(U)) = \deg(\psi, U)$$

and since the vector field  $\psi$  is tangent to  $\partial U$  at any point, we get (see, for example, Theorem 2.3 of [18])

$$\deg(\psi, U) = 1.$$

□

The following result is a direct consequence of Lemma 2.

**Corollary 1** *Under condition  $(A_0)$  we have that*

$$\deg(I - \Omega(T, 0, \cdot), W_\gamma(U)) = 1, \text{ for every } -\gamma_0 < \gamma < \gamma_0, \gamma \neq 0. \quad (10)$$

**Proof:** First we prove (10) for  $0 < \gamma < \gamma_0$ . By lemma 2 there exists  $0 < \gamma_* < \gamma_0$  such that  $\deg(I - \Omega(T, 0, \cdot), W_{\gamma_*}(U)) = 1$ . From definition 1 it follows that the constant  $\gamma_0$  has the property that the vector field  $I - \Omega(T, 0, \cdot)$  is not degenerated on the boundary of the set  $W_\gamma(U)$  for any  $0 < \gamma < \gamma_0$ . Therefore, for any  $0 < \gamma < \gamma_0$ , we have

$$\deg(I - \Omega(T, 0, \cdot), W_\gamma(U)) = \deg(I - \Omega(T, 0, \cdot), W_{\gamma_*}(U)) = 1.$$

The same arguments apply for  $-\gamma_0 < \gamma < 0$ . □

We are now in the position to prove theorem 1.

**Proof:** Denote by  $C([0, T], \mathbb{R}^2)$  the Banach space of all the continuous functions defined on the interval  $[0, T]$  with values in  $\mathbb{R}^2$ , equipped with the sup-norm. Consider in (1) the change of variable

$$x(t) = \Omega(t, 0, z(t)). \quad (11)$$

For every  $z \in C([0, T], \mathbb{R}^2)$ , (11) defines uniquely  $x \in C([0, T], \mathbb{R}^2)$  with inverse given by

$$z(t) = \Omega(0, t, x(t)), \quad t \in [0, T]. \quad (12)$$

Therefore, the function  $x$  is the solution of the system (1) if and only if the function  $z$  defined by (12) satisfies the differential equation

$$\Omega'_{(1)}(t, 0, z(t)) + \Omega'_{(3)}(t, 0, z(t)) \dot{z}(t) = \varepsilon \phi(t, \Omega(t, 0, z(t))) + \psi(\Omega(t, 0, z(t))). \quad (13)$$

By the definition of  $\Omega(t, 0, z(t))$  we have

$$\Omega'_{(1)}(t, 0, z(t)) = \psi(\Omega(t, 0, z(t))). \quad (14)$$

Moreover, by using (6) and (14) we can rewrite system (13) in the following form

$$\dot{z}(t) = \varepsilon \Phi(t, z(t)), \quad (15)$$

where  $\Phi$  is defined by (5). Observe that (11) and (12) define a homeomorphism between the solutions of systems (1) and (15).

Consider an arbitrary  $T$ -periodic solution  $x$  of system (1), we have

$$z(0) = \Omega(0, 0, x(0)) = x(0) = x(T) = \Omega(T, 0, z(T)).$$

Therefore the problem of the existence of  $T$ -periodic solutions to system (1) is equivalent to the problem of the existence of zeros of the compact vector field  $G_\varepsilon : C([0, T], \mathbb{R}^2) \rightarrow C([0, T], \mathbb{R}^2)$  defined by

$$G_\varepsilon(z)(t) = z(t) - \Omega(T, 0, z(T)) - \varepsilon \int_0^t \Phi(\tau, z(\tau)) d\tau, \quad t \in [0, T].$$

Define the set

$$Z = \left\{ z \in C([0, T], \mathbb{R}^2) : z(t) \in B_\gamma(\partial U) \text{ for any } t \in [0, T] \right\}. \quad (16)$$

Consider the auxiliary compact vector field

$$G_{1,\varepsilon} = I - A_\varepsilon : C([0, T], \mathbb{R}^2) \rightarrow C([0, T], \mathbb{R}^2),$$

where

$$A_\varepsilon(z)(t) = \Omega(T, 0, z(T)) - \varepsilon \int_0^T \Phi(\tau, z(\tau)) d\tau, \quad t \in [0, T],$$

Let us show that for any  $\varepsilon > 0$  satisfying (8) the compact vector fields  $G_\varepsilon$  and  $G_{1,\varepsilon}$  are homotopic on the boundary of the set  $Z$ . For this, define the following homotopy  $F_\varepsilon : [0, 1] \times C([0, T], \mathbb{R}^2) \rightarrow C([0, T], \mathbb{R}^2)$  joining the vector fields  $G_\varepsilon$  and  $G_{1,\varepsilon}$ :

$$F_\varepsilon(\lambda, z)(t) = z(t) - \Omega(T, 0, z(T)) - \varepsilon \int_0^{\alpha(\lambda, t)} \Phi(\tau, z(\tau)) d\tau, \quad t \in [0, T],$$

where  $\alpha(\lambda, t) = \lambda t + (1 - \lambda)T$ . Let us show that for any  $\varepsilon > 0$  satisfying (8) the homotopy  $F_\varepsilon$  does not vanish on the boundary of the set  $Z$ . Assume the contrary, thus for some  $\varepsilon > 0$  satisfying (8) there exist  $z_\varepsilon \in \partial Z$  and  $\lambda_\varepsilon \in [0, 1]$  such that

$$z_\varepsilon(t) = \Omega(T, 0, z_\varepsilon(T)) + \varepsilon \int_0^{\alpha(\lambda_\varepsilon, t)} \Phi(\tau, z_\varepsilon(\tau)) d\tau, \quad t \in [0, T]. \quad (17)$$

From the fact that  $z_\varepsilon \in \partial Z$  it follows the existence of  $t_\varepsilon \in [0, T]$  such that  $z_\varepsilon(t_\varepsilon) \in \partial(B_\gamma(\partial U))$ . By the definition of the set  $B_\gamma(\partial U)$  either

$$z_\varepsilon(t_\varepsilon) \in \partial W_\gamma(U), \quad (18)$$

or

$$z_\varepsilon(t_\varepsilon) \in \partial U. \quad (19)$$

By using (8) we have the following estimate

$$\begin{aligned} & \|z_\varepsilon(t_\varepsilon) - \Omega(T, 0, z_\varepsilon(t_\varepsilon))\| = \\ & \|z_\varepsilon(t_\varepsilon) - \Omega(T, 0, z_\varepsilon(T)) + \Omega(T, 0, z_\varepsilon(T)) - \Omega(T, 0, z_\varepsilon(t_\varepsilon))\| = \\ & = \left\| \varepsilon \int_0^{\alpha(\lambda_\varepsilon, t_\varepsilon)} \Phi(\tau, z_\varepsilon(\tau)) d\tau + \Omega(T, 0, z_\varepsilon(T)) - \Omega(T, 0, z_\varepsilon(t_\varepsilon)) \right\| \leq \\ & \leq \varepsilon T M_\gamma + \varepsilon L'_\gamma T M_\gamma < K_\gamma, \end{aligned} \quad (20)$$

which contradicts the definition of the constant  $K_\gamma$  in the case when (18) holds true. Thus (18) cannot occur. Assume now (19), by lemma 1 and the fact that, in this case,  $z_\varepsilon(t_\varepsilon) = \Omega(T, 0, z_\varepsilon(t_\varepsilon))$  we have

$$\begin{aligned} & \eta(T, \alpha(\lambda_\varepsilon, t_\varepsilon), z_\varepsilon(t_\varepsilon)) - \eta(0, \alpha(\lambda_\varepsilon, t_\varepsilon), z_\varepsilon(t_\varepsilon)) = \\ & = \left( \Omega'_{(3)}(T, 0, z_\varepsilon(t_\varepsilon)) - I \right) \int_{\alpha(\lambda_\varepsilon, t_\varepsilon)}^T \Phi(\tau, z_\varepsilon(t_\varepsilon)) d\tau + \int_0^T \Phi(\tau, z_\varepsilon(t_\varepsilon)) d\tau = \end{aligned}$$



$$\begin{aligned}
&= \frac{z_\varepsilon(T) - z_\varepsilon(t_\varepsilon)}{\varepsilon} - \frac{\Omega(T, 0, z_\varepsilon(T)) - \Omega(T, 0, z_\varepsilon(t_\varepsilon))}{\varepsilon} - \int_0^T \Phi(\tau, z_\varepsilon(\tau)) d\tau - \\
&- \left( I - \Omega'_{(3)}(T, 0, z_\varepsilon(t_\varepsilon)) \right) \int_{\alpha(\lambda_\varepsilon, t_\varepsilon)}^T \Phi(\tau, z_\varepsilon(t_\varepsilon)) d\tau + \int_0^T \Phi(\tau, z_\varepsilon(t_\varepsilon)) d\tau = \\
&= \int_{\alpha(\lambda_\varepsilon, t_\varepsilon)}^T \Phi(\tau, z_\varepsilon(\tau)) d\tau - \frac{1}{\varepsilon} \begin{pmatrix} [\Omega'_{(3)}(T, 0, \xi_{1\varepsilon})(z_\varepsilon(T) - z_\varepsilon(t_\varepsilon))]_1 \\ [\Omega'_{(3)}(T, 0, \xi_{2\varepsilon})(z_\varepsilon(T) - z_\varepsilon(t_\varepsilon))]_2 \end{pmatrix} - \\
&- \left( I - \Omega'_{(3)}(T, 0, z_\varepsilon(t_\varepsilon)) \right) \int_{\alpha(\lambda_\varepsilon, t_\varepsilon)}^T \Phi(\tau, z_\varepsilon(t_\varepsilon)) d\tau + \int_0^T \Phi(\tau, z_\varepsilon(t_\varepsilon)) d\tau - \\
&- \int_0^T \Phi(\tau, z_\varepsilon(\tau)) d\tau = \int_0^{\alpha(\lambda_\varepsilon, t_\varepsilon)} \Phi(\tau, z_\varepsilon(t_\varepsilon)) d\tau - \int_0^{\alpha(\lambda_\varepsilon, t_\varepsilon)} \Phi(\tau, z_\varepsilon(\tau)) d\tau - \\
&- \begin{pmatrix} [\Omega'_{(3)}(T, 0, \xi_{1\varepsilon}) \int_{\alpha(\lambda_\varepsilon, t_\varepsilon)}^T \Phi(\tau, z_\varepsilon(\tau)) d\tau]_1 \\ [\Omega'_{(3)}(T, 0, \xi_{2\varepsilon}) \int_{\alpha(\lambda_\varepsilon, t_\varepsilon)}^T \Phi(\tau, z_\varepsilon(\tau)) d\tau]_2 \end{pmatrix} + \\
&+ \begin{pmatrix} [\Omega'_{(3)}(T, 0, z_\varepsilon(t_\varepsilon)) \int_{\alpha(\lambda_\varepsilon, t_\varepsilon)}^T \Phi(\tau, z_\varepsilon(\tau)) d\tau]_1 \\ [\Omega'_{(3)}(T, 0, z_\varepsilon(t_\varepsilon)) \int_{\alpha(\lambda_\varepsilon, t_\varepsilon)}^T \Phi(\tau, z_\varepsilon(\tau)) d\tau]_2 \end{pmatrix} - \\
&- \Omega'_{(3)}(T, 0, z_\varepsilon(t_\varepsilon)) \int_{\alpha(\lambda_\varepsilon, t_\varepsilon)}^T \Phi(\tau, z_\varepsilon(\tau)) d\tau + \Omega'_{(3)}(T, 0, z_\varepsilon(t_\varepsilon)) \int_{\alpha(\lambda_\varepsilon, t_\varepsilon)}^T \Phi(\tau, z_\varepsilon(t_\varepsilon)) d\tau, \quad (21)
\end{aligned}$$

where  $\xi_{i\varepsilon} \in [z_\varepsilon(t_\varepsilon), z_\varepsilon(T)]_i$  and  $[v]_i, i = 1, 2$ , denotes the  $i$ -th component of the vector  $v$ . By using the following estimates

$$\begin{aligned}
\|z_\varepsilon(t_\varepsilon) - \xi_{i\varepsilon}\| &\leq \|z_\varepsilon(t_\varepsilon) - z_\varepsilon(T)\| \leq \left\| \varepsilon \int_{\alpha(\lambda_\varepsilon, t_\varepsilon)}^T \Phi(\tau, z_\varepsilon(\tau)) d\tau \right\| \leq \varepsilon T M_\gamma \quad \text{and} \\
\max_{t \in [0, T]} \|\Phi(t, z_\varepsilon(t_\varepsilon)) - \Phi(t, z_\varepsilon(t))\| &\leq M'_\gamma \max_{t \in [0, T]} \|z_\varepsilon(t_\varepsilon) - z_\varepsilon(t)\| \leq \varepsilon T M_\gamma M'_\gamma
\end{aligned}$$

together with (8) we obtain from (21)

$$\begin{aligned}
&\|\eta(T, \alpha(\lambda_\varepsilon, t_\varepsilon), z_\varepsilon(t_\varepsilon)) - \eta(0, \alpha(\lambda_\varepsilon, t_\varepsilon), z_\varepsilon(t_\varepsilon))\| \leq \\
&\leq \varepsilon T^2 M_\gamma M'_\gamma + \varepsilon \sqrt{2} T^2 (M_\gamma)^2 L''_\gamma + \varepsilon T^2 M_\gamma M'_\gamma L'_\gamma = \\
&= \varepsilon T^2 M_\gamma (M'_\gamma + \sqrt{2} M_\gamma L''_\gamma + M'_\gamma L'_\gamma) < K_0, \quad (22)
\end{aligned}$$

which contradicts the definition of the constant  $K_0$ . Hence both (18) and (19) cannot occur and so for  $\varepsilon \in (0, \varepsilon_\gamma)$  the homotopy  $F_\varepsilon$  does not vanish on the boundary of the set  $Z$  and so the compact vector

fields  $G_\varepsilon$  and  $G_{1,\varepsilon}$  are homotopic on the boundary of the set  $Z$ . In particular,

$$G_{1,\varepsilon}(z) \neq 0, \text{ for any } z \in \partial \left( Z \cap C_{const}([0, T], \mathbb{R}^2) \right) \subset \partial Z, \text{ and any } \varepsilon \in (0, \varepsilon_\gamma), \quad (23)$$

where  $C_{const}([0, T], \mathbb{R}^2)$  denotes the subspace of the space  $C([0, T], \mathbb{R}^2)$  consisting of all the constant functions defined on the interval  $[0, T]$  with values in  $\mathbb{R}^2$  and  $\partial \left( Z \cap C_{const}([0, T], \mathbb{R}^2) \right)$  is the relative boundary of the set  $Z \cap C_{const}([0, T], \mathbb{R}^2)$  with respect to the subspace  $C_{const}([0, T], \mathbb{R}^2)$ . Therefore, by the reduction domain property of the topological degree, taking into account that  $A_\varepsilon(\partial Z) \subset C_{const}([0, T], \mathbb{R}^2)$ , we obtain

$$\deg_{C([0, T], \mathbb{R}^2)}(G_{1,\varepsilon}, Z) = \deg_{C_{const}([0, T], \mathbb{R}^2)} \left( G_{1,\varepsilon}, Z \cap C_{const}([0, T], \mathbb{R}^2) \right), \quad (24)$$

whenever  $\varepsilon \in (0, \varepsilon_\gamma)$ . Observe, that  $z \in Z \cap C_{const}([0, T], \mathbb{R}^2)$  is a solution of the equation  $G_{1,\varepsilon}z = 0$  if and only if  $\xi = z$  is a solution of the equation  $Q_\varepsilon \xi = 0$ , where  $Q_\varepsilon : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is defined by

$$Q_\varepsilon \xi = \xi - \Omega(T, 0, \xi) - \varepsilon \int_0^T \Phi(\tau, \xi) d\tau.$$

Therefore, we have

$$\begin{aligned} \deg_{C_{const}([0, T], \mathbb{R}^2)}(G_{1,\varepsilon}, Z \cap C_{const}([0, T], \mathbb{R}^2)) &= \deg_{\mathbb{R}^2}(Q_\varepsilon, B_\gamma(\partial U)) = \\ &= \text{sign}(\gamma) (\deg_{\mathbb{R}^2}(Q_\varepsilon, W_\gamma(U)) - \deg_{\mathbb{R}^2}(Q_\varepsilon, U)), \text{ for any } \varepsilon \in (0, \varepsilon_\gamma). \end{aligned} \quad (25)$$

From (8), for any  $\varepsilon \in (0, \varepsilon_\gamma)$ , we have

$$\min_{\xi \in \partial W_\gamma(U)} \|\xi - \Omega(T, 0, \xi)\| = K_\gamma > \varepsilon T M_\gamma \geq \varepsilon \max_{\xi \in B_\gamma(\partial U)} \left\| \int_0^T \Phi(\tau, \xi) d\tau \right\|$$

and so

$$\deg_{\mathbb{R}^2}(Q_\varepsilon, W_\gamma(U)) = \deg_{\mathbb{R}^2}(Q_0, W_\gamma(U)), \quad \text{whenever } \varepsilon \in (0, \varepsilon_\gamma).$$

By corollary 1 and the previous equality we obtain

$$\deg_{\mathbb{R}^2}(Q_\varepsilon, W_\gamma(U)) = 1, \quad \text{whenever } \varepsilon \in (0, \varepsilon_\gamma). \quad (26)$$

Let us now calculate  $\deg_{\mathbb{R}^2}(Q_\varepsilon, U)$ . For this, let  $Q_{1,\varepsilon} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined by

$$Q_{1,\varepsilon} \xi = -\varepsilon \int_0^T \Phi(\tau, \xi) d\tau, \quad \text{for any } \xi \in Z \cap C_{const}([0, T], \mathbb{R}^2).$$

From (A<sub>2</sub>) we have that  $Q_\varepsilon \xi = Q_{1,\varepsilon} \xi$ , for any  $\xi \in \partial U$ , since  $\xi = \Omega(T, 0, \xi)$ , and so for  $\varepsilon \in (0, \varepsilon_\gamma)$  we have

$$\deg_{\mathbb{R}^2}(Q_\varepsilon, U) = \deg_{\mathbb{R}^2}(Q_{1,\varepsilon}, U). \quad (27)$$

Let us show that the compact vector fields  $Q_{1,\varepsilon}$  and  $Q_{1,1}$  are homotopic on the boundary of the set  $U$  for  $\varepsilon \in (0, \varepsilon_\gamma)$ . For this, define the linear homotopy  $F_{1,\varepsilon} : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$  as follows

$$F_{1,\varepsilon}(\lambda, \xi) = -(\lambda\varepsilon + 1 - \lambda) \int_0^T \Phi(\tau, \xi) d\tau.$$

Arguing by contradiction, assume that for some  $\hat{\lambda} \in [0, 1]$ ,  $\hat{\xi} \in \partial U$  and  $\hat{\varepsilon} \in (0, \varepsilon_\gamma)$  we have

$$(\hat{\lambda}\hat{\varepsilon} + 1 - \hat{\lambda}) \int_0^T \Phi(\tau, \hat{\xi}) d\tau = 0,$$

and so

$$\int_0^T \Phi(\tau, \hat{\xi}) d\tau = 0.$$

By lemma 1 this is a contradiction with condition  $(A_2)$ . Thus, again by lemma 1

$$\deg_{\mathbb{R}^2}(Q_{1,\varepsilon}, U) = \deg_{\mathbb{R}^2}(Q_{1,1}, U) = \deg_{\mathbb{R}^2}(\eta(0, T, \cdot), U). \quad (28)$$

From (27) and (28) we obtain

$$\deg_{\mathbb{R}^2}(Q_\varepsilon, U) = \deg_{\mathbb{R}^2}(\eta(0, T, \cdot), U), \text{ for any } \varepsilon \in (0, \varepsilon_\gamma). \quad (29)$$

Finally, taking into account (24)-(26), (29) and condition  $(A_3)$  we get

$$\begin{aligned} \deg_{C([0,T], \mathbb{R}^2)}(G_\varepsilon, Z) &= \deg_{C([0,T], \mathbb{R}^2)}(G_{1,\varepsilon}, Z) = \\ &= \text{sign}(\gamma) (1 - \deg_{\mathbb{R}^2}(\eta(0, T, \cdot), U)) \neq 0, \end{aligned}$$

for any  $\varepsilon \in (0, \varepsilon_\gamma)$ . The solution property of the topological degree ends the proof.  $\square$

**Remark 1.** Observe that  $\deg(\eta(0, T, \cdot), U)$  can be calculated by means of a well known formula (see e. g. [17], p. 6). In this case condition  $(A_3)$  takes the form

$$\begin{aligned} \deg(\eta(0, T, \cdot), U) &= \sum_{i=0}^{n-1} \text{sign}([\eta(T, 0, x_0(\cdot))]_1, \theta_{i+1}, \theta_i) \cdot \\ &\quad \cdot (\text{sign}[\eta(T, 0, x_0(\theta_{i+1}))]_2 - \text{sign}[\eta(T, 0, x_0(\theta_i))]_2) \neq 1, \end{aligned}$$

where  $[\cdot]_j$ ,  $j = 1, 2$ , denotes the  $j$ -th component of the vector field  $\eta$  and  $\theta_i$ ,  $i = 0, 1, \dots, n-1$ ,  $\theta_n = \theta_0$ , are the ordered roots of function  $\theta \mapsto [\eta(0, T, x_0(\theta))]_1$ .

Conditions for the existence of solutions to (1) in terms of the derivatives of the function  $[\eta(0, T, x_0(\cdot))]_1$  at the points  $\theta_i$ , up to a suitable rotation, are given in [20].

We now prove the following result.

**Theorem 2** Assume that there exists  $\xi \in \tilde{x}_0$  such that  $T_1 > 0$  is the least period of the function  $t \rightarrow \phi(t, \xi)$ . Assume condition  $(A_1)$ , if  $\{x_\varepsilon\}_{\varepsilon \in (0, \varepsilon_*)}$  are  $\hat{T}$ -periodic solutions to (1) converging to  $\tilde{x}_0$  as  $\varepsilon \rightarrow 0$ , then  $klT_0 \leq \hat{T}$ .

**Proof:** Assume the contrary, thus there exists  $\tilde{T} \in (0, klT_0)$  such that for every  $\varepsilon > 0$  sufficiently small system (1) has a  $\tilde{T}$ -periodic solution  $x_\varepsilon$  satisfying

$$\lim_{\varepsilon \rightarrow 0} x_\varepsilon(t) = x_0(t + w_0),$$

for some  $w_0 \in [0, T_0]$ , thus  $\tilde{T}$  is a period of the function  $x_0(t)$ . On the other hand,  $T_0$  is the least period of  $x_0(t)$  and so we obtain  $\tilde{T} = n_0 T_0$  for some  $n_0 \in \mathbb{N}$ . Moreover, in (1) we have that the function  $t \rightarrow \phi(t, x_\varepsilon(t))$  is  $\tilde{T}$ -periodic, since  $\dot{x}(t)$  is  $\tilde{T}$ -periodic, and so for any  $t_0 \in [0, T]$ , we have

$$\phi(t_0, x_\varepsilon(t_0)) = \phi(t_0 + \tilde{T}, x_\varepsilon(t_0 + \tilde{T})) = \phi(t_0 + \tilde{T}, x_\varepsilon(t_0)).$$

Due to our assumption and the arbitrariness of  $t_0$ , by passing to the limit as  $\varepsilon \rightarrow 0$ , we conclude that  $\tilde{T} = n_1 T_1$  for some  $n_1 \in \mathbb{N}$ . Therefore

$$n_0 T_0 = n_1 T_1$$

and so by  $(A_1)$  there exists  $n_2 \in \mathbb{N}$  such that  $n_0 = kln_2$ , contradicting the fact that  $\tilde{T} < klT_0$ .  $\square$

### 3 A method to verify assumptions $(A_2)$ and $(A_3)$ of theorem 1

Although the topological degree (or rotation number) of the vector field in  $(A_3)$  can be calculated by using the methods from [17], as recalled in remark 1, we provide here an alternative method to verify conditions  $(A_2)$  and  $(A_3)$  which turns out to be useful and of simple application.

First, we introduce some notations. Recall that for the vector  $v \in \mathbb{R}^2$  we denote by  $[v]_i$  its  $i$ -th component,  $i = 1, 2$ , for  $a \in \mathbb{R}^2$  we put

$$a^\perp = \begin{pmatrix} -[a]_2 \\ [a]_1 \end{pmatrix}, \quad a^\top = \begin{pmatrix} [a]_2 \\ -[a]_1 \end{pmatrix},$$

and for  $a, b \in \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} [a]_1 & [a]_2 \\ [b]_1 & [b]_2 \end{pmatrix}, \quad (a \ b) = \begin{pmatrix} [a]_1 & [b]_1 \\ [a]_2 & [b]_2 \end{pmatrix}.$$

Moreover,  $y = y(t)$  will denote the solution of system (3) linearly independent with  $\dot{x}_0(t)$  and  $(s, \theta) \rightarrow F(s, \theta)$  is the following function

$$F(s, \theta) = \int_{s-T}^s (\dot{x}_0(\tau) \ y(\tau))^{-1} \phi(\tau - \theta, x_0(\tau)) d\tau.$$

We can prove the following.

**Theorem 3** *Let  $f : [0, T_0] \rightarrow \mathbb{R}^2$  be a function satisfying the following conditions*

$(B_2)$   $\langle F(s, \theta), f(\theta) \rangle \neq 0$ , for any  $s \in [0, T]$ ,  $\theta \in [0, T_0]$ ,

$(B_3)$   $\deg(N, [0, T_0]) \neq 1$ ,

where  $N(\theta) = \left( y(\theta)^\top \dot{x}_0(\theta)^\perp \right) f(\theta)$ ,  $\theta \in [0, T_0]$ . Then assumptions  $(A_2)$  and  $(A_3)$  of theorem 1 are satisfied.

To prove this theorem we need the following preliminary lemma.

**Lemma 3** *We have that*

$$\begin{aligned} & \langle \eta(T, s, x_0(\theta)) - \eta(0, s, x_0(\theta)), N(\theta) \rangle = \\ &= \int_{s-T+\theta}^{s+\theta} \left\langle d(\tau, 0) \phi(\tau - \theta, x_0(\tau)), \begin{matrix} < \dot{x}_0(\theta), N(\theta) > y(\tau)^\top + < y(\theta), N(\theta) > \dot{x}_0(\tau)^\perp \end{matrix} \right\rangle d\tau, \end{aligned} \quad (30)$$

where

$$d(t, \theta) = \left( \det \begin{pmatrix} y(t + \theta)^\top \\ \dot{x}_0(t + \theta)^\perp \end{pmatrix} \right)^{-1}.$$

**Proof:** Observe, that

$$\Omega'_{(3)}(0, t, \Omega(t, 0, x_0(\theta))) = K(0, \theta) (K(t, \theta))^{-1},$$

where  $K(\cdot, \theta)$  is a matrix whose columns are linearly independent solutions of the differential system

$$\dot{x} = \psi'(x_0(t + \theta))x. \quad (31)$$

Let us choose the matrix  $K(\cdot, \theta)$  as follows

$$K(t, \theta) = \begin{pmatrix} \dot{x}_0(t + \theta) & y(t + \theta) \end{pmatrix}.$$

Thus

$$\begin{aligned} \Omega'_{(3)}(0, t, \Omega(t, 0, x_0(\theta))) &= \begin{pmatrix} \dot{x}_0(\theta) & y(\theta) \end{pmatrix} \begin{pmatrix} \dot{x}_0(t + \theta) & y(t + \theta) \end{pmatrix}^{-1} = \\ &= \begin{pmatrix} \dot{x}_0(\theta) & y(\theta) \end{pmatrix} \begin{pmatrix} y(t + \theta)^\top \\ \dot{x}_0(t + \theta)^\perp \end{pmatrix} d(\tau, \theta). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} & \langle \eta(T, s, x_0(\theta)) - \eta(0, s, x_0(\theta)), N(\theta) \rangle = \\ &= \left\langle \int_{s-T}^s \Omega'_{(3)}(0, \tau, \Omega(\tau, 0, x_0(\theta))) \phi(\tau, x_0(\tau + \theta)) d\tau, N(\theta) \right\rangle = \\ &= \left\langle \int_{s-T}^s d(\tau, \theta) \begin{pmatrix} \dot{x}_0(\theta) & y(\theta) \end{pmatrix} \begin{pmatrix} y(\tau + \theta)^\top \\ \dot{x}_0(\tau + \theta)^\perp \end{pmatrix} \phi(\tau, x_0(\tau + \theta)) d\tau, N(\theta) \right\rangle = \\ &= \left\langle \int_{s-T}^s d(\tau, \theta) \begin{pmatrix} y(\tau + \theta)^\top \\ \dot{x}_0(\tau + \theta)^\perp \end{pmatrix} \phi(\tau, x_0(\tau + \theta)) d\tau, \begin{pmatrix} \dot{x}_0(\theta) \\ y(\theta) \end{pmatrix} N(\theta) \right\rangle = \\ &= \left\langle \int_{s-T}^s d(\tau, \theta) \begin{pmatrix} y(\tau + \theta)^\top \\ \dot{x}_0(\tau + \theta)^\perp \end{pmatrix} \phi(\tau, x_0(\tau + \theta)) d\tau, \begin{pmatrix} < \dot{x}_0(\theta), N(\theta) > \\ < y(\theta), N(\theta) > \end{pmatrix} \right\rangle = \\ &= \int_{s-T}^s \left\langle d(\tau, \theta) \phi(\tau, x_0(\tau + \theta)), (y(\tau + \theta)^\top \quad \dot{x}_0(\tau + \theta)^\perp) \begin{pmatrix} < \dot{x}_0(\theta), N(\theta) > \\ < y(\theta), N(\theta) > \end{pmatrix} \right\rangle d\tau = \end{aligned}$$

$$\begin{aligned}
&= \int_{s-T}^s \left\langle d(\tau, \theta) \phi(\tau, x_0(\tau + \theta)), \langle \dot{x}_0(\theta), N(\theta) \rangle y(\tau + \theta)^\top + \langle y(\theta), N(\theta) \rangle \dot{x}_0(\tau + \theta)^\perp \right\rangle d\tau = \\
&= \int_{s-T+\theta}^{s+\theta} \left\langle d(\tau, 0) \phi(\tau - \theta, x_0(\tau)), \langle \dot{x}_0(\theta), N(\theta) \rangle y(\tau)^\top + \langle y(\theta), N(\theta) \rangle \dot{x}_0(\tau)^\perp \right\rangle d\tau.
\end{aligned}$$

□

We can now prove Theorem 3.

**Proof:** By lemma 3 and the fact that

$$\left\langle \dot{x}_0(\theta), y(\theta)^\top \right\rangle = \left\langle y(\theta), \dot{x}_0(\theta)^\perp \right\rangle,$$

we have

$$\begin{aligned}
&\langle \eta(T, s, x_0(\theta)) - \eta(0, s, x_0(\theta)), N(\theta) \rangle = \\
&\int_{s-T+\theta}^{s+\theta} \left\langle d(\tau, 0) \phi(\tau - \theta, x_0(\tau)), [f(\theta)]_1 \left\langle \dot{x}_0(\theta), y(\theta)^\top \right\rangle y(\tau)^\top + \right. \\
&+ [f(\theta)]_2 \left\langle \dot{x}_0(\theta), \dot{x}_0(\theta)^\perp \right\rangle y(\tau)^\top + [f(\theta)]_1 \left\langle y(\theta), y(\theta)^\top \right\rangle \dot{x}_0(\tau)^\perp + \\
&+ [f(\theta)]_2 \left\langle y(\theta), \dot{x}_0(\theta)^\perp \right\rangle \dot{x}_0(\tau)^\perp \left. \right\rangle d\tau = \\
&= \int_{s-T+\theta}^{s+\theta} \left\langle d(\tau, 0) \phi(\tau - \theta, x_0(\tau)), \left\langle \dot{x}_0(\theta), y(\theta)^\top \right\rangle \left( [f(\theta)]_1 y(\tau)^\top + [f(\theta)]_2 \dot{x}_0(\tau)^\perp \right) \right\rangle d\tau = \\
&= \left\langle \dot{x}_0(\theta) y(\theta)^\top \right\rangle \int_{s-T+\theta}^{s+\theta} \left\langle d(\tau, 0) \phi(\tau - \theta, x_0(\tau)), \left( y(\tau)^\top \dot{x}_0(\tau)^\perp \right) f(\theta) \right\rangle d\tau = \\
&= \left\langle \dot{x}_0(\theta) y(\theta)^\top \right\rangle \int_{s-T+\theta}^{s+\theta} \left\langle d(\tau, \theta) \begin{pmatrix} y(\tau)^\top \\ \dot{x}_0(\tau)^\perp \end{pmatrix} \phi(\tau - \theta, x_0(\tau)), f(\theta) \right\rangle d\tau.
\end{aligned}$$

Therefore, by the condition (B<sub>2</sub>)

$$\langle \eta(T, s, x_0(\theta)) - \eta(0, s, x_0(\theta)), N(\theta) \rangle \neq 0, \quad \text{for any } s \in [0, T], \theta \in [0, T_0], \quad (32)$$

that is the condition (A<sub>2</sub>) of Theorem 1 hold true. Condition (A<sub>3</sub>) of theorem 1 also follows from (32) by taking into account condition (B<sub>3</sub>) and lemma 1. □

#### 4 The case when $\frac{T_0}{T_1}$ is irrational

In this section we assume the following condition

(A'<sub>1</sub>)-  $\frac{T_0}{T_1}$  is an irrational number.

We can prove the following result.

**Theorem 4** *Assume that there exists  $\xi \in \tilde{x}_0$  such that the function  $\phi(t, \xi)$  is not constant with respect to  $t$ . Assume  $(A'_1)$ , then system (1) does not have  $T$ -periodic solutions  $x_\varepsilon$ ,  $\varepsilon \in (0, \hat{\varepsilon})$ , converging to  $\tilde{x}_0$  as  $\varepsilon \rightarrow 0$ , whenever  $T > 0$  and  $\hat{\varepsilon} > 0$ .*

**Proof:** Assume the contrary, thus there exists  $T > 0$  and  $\hat{\varepsilon} > 0$  such that for every  $\varepsilon \in (0, \hat{\varepsilon})$  system (1) has  $T$ -periodic solution  $x_\varepsilon(t)$  satisfying

$$\lim_{\varepsilon \rightarrow 0} x_\varepsilon(t) = x_0(t + w_0),$$

for some  $w_0 \in [0, T_0]$ . Hence  $T$  is a period for the function  $x_0$ . On the other hand, since  $T_0$  is the least period of  $x_0$ , we have that  $T = n_0 T_0$  for some  $n_0 \in \mathbb{N}$ . Moreover in (1) we have that the function  $t \rightarrow \phi(t, x_\varepsilon(t))$  is  $T$ -periodic, since  $\dot{x}(t)$  is  $T$ -periodic, thus, for any  $t_0 \in [0, T]$ , we get

$$\phi(t_0, x_\varepsilon(t_0)) = \phi(t_0 + p n_0 T_0, x_\varepsilon(t_0 + p n_0 T_0)) = \phi(t_0 + T_p, x_\varepsilon(t_0)), \text{ whenever } p \in \mathbb{N},$$

where  $T_p = p n_0 T_0 \pmod{T_1}$ . Condition  $(A'_1)$  implies that

$$\overline{\bigcup_{p \in \mathbb{N}} T_p} = [0, T_1]$$

and so

$$\phi(t, x_\varepsilon(t_0)) = c_\varepsilon, \text{ for any } t \in [0, T_1],$$

with  $c_\varepsilon \in \mathbb{R}^2$ . By letting  $\varepsilon \rightarrow 0$  in the previous relation we obtain a contradiction with the assumption that  $t \rightarrow \phi(t, \xi)$  is not constant for at least one  $\xi \in \tilde{x}_0$ , in fact  $t_0$  is any point of  $[0, T]$ , and  $T = n_0 T_0, n_0 \in \mathbb{N}$ .  $\square$

## 5 Example

In this section we consider the following illustrative example for system (2).

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 &= -x_1 - x_2(x_1^2 + x_2^2 - 1). \end{aligned} \tag{33}$$

It is easy to see that system (33) has the limit circle  $x_0(\theta) = (\sin \theta, \cos \theta)$  with period  $T_0 = 2\pi$ . In order to verify the conditions of theorem 1 we use theorem 3. Thus, we consider

$$\dot{x}_0(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix},$$

It is easy to verify that  $y(t) = e^{-2t}(\sin t, \cos t)$  satisfies the linearized system (3) corresponding to system (33). Therefore

$$\begin{aligned} y(t)^\perp &= e^{-2t} \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix} \\ y(t)^\top &= e^{-2t} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}. \end{aligned}$$

Define the function  $f : [0, 2\pi] \rightarrow (\mathbb{R})^2$  by the formula

$$f(\theta) = \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix}.$$

If  $t \rightarrow \phi(t, x)$  is a  $4\pi$ -periodic function, then condition (B<sub>2</sub>) of theorem 3 takes the form

$$\left\langle \int_{s-4\pi+\theta}^{s+\theta} \begin{pmatrix} \cos \tau & e^{-2\tau} \sin \tau \\ -\sin \tau & e^{-2\tau} \cos \tau \end{pmatrix}^{-1} \phi(\tau - \theta, x_0(\tau)) d\tau, \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix} \right\rangle \neq 0 \quad (34)$$

for any  $s \in [0, 4\pi]$ ,  $\theta \in [0, 2\pi]$ .

As example of function  $\phi$  satisfying condition (34) we consider

$$\phi(t, \xi) = \begin{pmatrix} \xi_2 & \xi_1 \\ -\xi_1 & \xi_2 \end{pmatrix} \begin{pmatrix} \xi_1 \cos t - \xi_2 \sin t + a \sin \frac{t}{2} \\ \xi_1 \sin t + \xi_2 \cos t \end{pmatrix}, \quad (35)$$

with  $a > 0$  sufficiently small. In fact, for  $a = 0$  we have

$$\begin{aligned} & \left\langle \int_{s-4\pi+\theta}^{s+\theta} \begin{pmatrix} \cos \tau & e^{-2\tau} \sin \tau \\ -\sin \tau & e^{-2\tau} \cos \tau \end{pmatrix}^{-1} \phi(\tau - \theta, x_0(\tau)) d\tau, \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix} \right\rangle = \\ & = \left\langle \int_{s-4\pi+\theta}^{s+\theta} \begin{pmatrix} \cos \tau & -\sin \tau \\ e^{2\tau} \sin \tau & e^{2\tau} \cos \tau \end{pmatrix} \begin{pmatrix} \cos \tau & \sin \tau \\ -\sin \tau & \cos \tau \end{pmatrix} \cdot \begin{pmatrix} \sin \tau \cos(\tau - \theta) - \cos \tau \sin(\tau - \theta) \\ \sin \tau \sin(\tau - \theta) + \cos \tau \cos(\tau - \theta) \end{pmatrix} d\tau, \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix} \right\rangle \\ & = \left\langle (\sin \theta \cos \theta) \int_{s-4\pi+\theta}^{s+\theta} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\tau} \end{pmatrix} d\tau, \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix} \right\rangle = \\ & = 4\pi \sin^2 \theta + \cos^2 \theta \int_{s-4\pi+\theta}^{s+\theta} e^{2\tau} d\tau > 0 \end{aligned} \quad (36)$$

and so (34) holds for the function (35) for any  $a > 0$  sufficiently small.

Let us now verify condition (B<sub>3</sub>) of theorem 3. For this, consider

$$N(\theta) = \begin{pmatrix} e^{-2\theta} \cos \theta & \sin \theta \\ -e^{-2\theta} \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix}.$$

and observe that the homotopy

$$N_\lambda(\theta) = \begin{pmatrix} e^{-2\lambda\theta} \cos \theta & \sin \theta \\ -e^{-2\lambda\theta} \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix},$$

joining the vector fields  $N_0$  and  $N_1 = N$  does not vanish whenever  $\lambda \in [0, 1]$ . Therefore

$$\deg(N, [0, 2\pi]) = \deg(N_0, [0, 2\pi]) = \deg \left( \begin{pmatrix} \sin(2\cdot) \\ \cos(2\cdot) \end{pmatrix}, [0, 2\pi] \right) = 2$$



and so condition (B<sub>3</sub>) of theorem 3 is also satisfied.

In conclusion, for  $0 < \varepsilon < \varepsilon_\gamma$ , where  $\varepsilon_\gamma > 0$  is given as in (8), the following system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 - x_1(x_1^2 + x_2^2 - 1) \\ -x_1 - x_2(x_1^2 + x_2^2 - 1) \end{pmatrix} + \\ + \varepsilon \begin{pmatrix} x_2 & x_1 \\ -x_1 & x_2 \end{pmatrix} \begin{pmatrix} x_1 \cos t - x_2 \sin t + a \sin \frac{t}{2} \\ x_1 \sin t + x_2 \cos t \end{pmatrix},$$

where  $a > 0$  is sufficiently small, has a  $4\pi$ -periodic solutions  $x_{\varepsilon,1}$  and  $x_{\varepsilon,2}$  which converge to the unitary circle from the outside and from the inside respectively as  $\varepsilon \rightarrow 0$ .

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